Exercise 1.1.4

Gauss' test is often given in the form of a test of the ratio

$$\frac{u_n}{u_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0}.$$

For what values of the parameters a_1 and b_1 is there convergence? divergence?

$$\textit{ANS.} \quad \text{Convergent for } a_1 - b_1 > 1, \qquad \text{divergent for } a_1 - b_1 \leq 1.$$

Solution

Invert both sides

$$\frac{u_{n+1}}{u_n} = \frac{n^2 + b_1 n + b_0}{n^2 + a_1 n + a_0}$$

and take the limit of both sides as $n \to \infty$.

$$\begin{aligned} \lim_{n\to\infty} \frac{u_{n+1}}{u_n} &= \lim_{n\to\infty} \frac{n^2 + b_1 n + b_0}{n^2 + a_1 n + a_0} \\ &= \lim_{n\to\infty} \frac{1 + \frac{b_1}{n} + \frac{b_0}{n^2}}{1 + \frac{a_1}{n} + \frac{a_0}{n^2}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

As a result, the ratio test is inconclusive. A more sensitive test is necessary to determine whether $\sum_n u_n$ is convergent or divergent. The Gauss test applies in this situation:

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2} \tag{1}$$

If $B(n)$ is bounded for large enough n, then $\sum_n u_n$ converges when $h > 1$ and diverges when $h \leq 1$. The goal, then, is to use long division to rewrite the right side of the given ratio.

$$n^2 + b_1 n + b_0 \;\overline{)\, n^2 + a_1 n + a_0}$$

Multiply n^2 by 1 to get n^2. Then subtract 1 times the divisor from the dividend.

$$\begin{array}{r} 1 \\ n^2 + b_1 n + b_0 \;\overline{)\, n^2 + a_1 n + a_0} \\ -(n^2 + b_1 n + b_0) \end{array}$$

Do the subtraction.

$$
\begin{array}{r|l}
 & 1 \\
\hline
n^2 + b_1 n + b_0 & n^2 + a_1 n + a_0 \\
 & -(n^2 + b_1 n + b_0) \\
\hline
 & (a_1 - b_1) n + (a_0 - b_0)
\end{array}
$$

Multiply n^2 by $(a_1 - b_1)/n$ to get $(a_1 - b_1)n$. Then subtract $(a_1 - b_1)/n$ times the divisor from the dividend.

$$
\begin{array}{r|l}
 & 1 + \dfrac{a_1 - b_1}{n} \\
\hline
n^2 + b_1 n + b_0 & n^2 + a_1 n + a_0 \\
 & -(n^2 + b_1 n + b_0) \\
\hline
 & (a_1 - b_1) n + (a_0 - b_0) \\
 & -\left[(a_1 - b_1) n + (a_1 - b_1) b_1 + \dfrac{(a_1 - b_1) b_0}{n}\right]
\end{array}
$$

Do the subtraction.

$$
\begin{array}{r l}
 & 1 + \frac{a_1 - b_1}{n} \\
\cline{2-2}
n^2 + b_1 n + b_0 & \big) \, n^2 + a_1 n + a_0 \\
 & -\left(n^2 + b_1 n + b_0\right) \\
\cline{2-2}
 & (a_1 - b_1) n + (a_0 - b_0) \\
 & -\left[(a_1 - b_1) n + (a_1 - b_1) b_1 + \frac{(a_1 - b_1) b_0}{n}\right] \\
\cline{2-2}
 & \left(a_0 - b_0 - a_1 b_1 + b_1{}^2\right) - \frac{(a_1 - b_1) b_0}{n}
\end{array}
$$

Multiply n^2 by $(a_0 - b_0 - a_1 b_1 + b_1^2)/n^2$ to get $a_0 - b_0 - a_1 b_1 + b_1^2$. Then subtract $(a_0 - b_0 - a_1 b_1 + b_1^2)/n^2$ times the divisor from the dividend.

$$
\begin{array}{r l}
 & 1 + \frac{a_1 - b_1}{n} + \frac{a_0 - b_0 - a_1 b_1 + b_1{}^2}{n^2} \\
\cline{2-2}
n^2 + b_1 n + b_0 & \big) \, n^2 + a_1 n + a_0 \\
 & -\left(n^2 + b_1 n + b_0\right) \\
\cline{2-2}
 & (a_1 - b_1) n + (a_0 - b_0) \\
 & -\left[(a_1 - b_1) n + (a_1 - b_1) b_1 + \frac{(a_1 - b_1) b_0}{n}\right] \\
\cline{2-2}
 & \left(a_0 - b_0 - a_1 b_1 + b_1{}^2\right) - \frac{(a_1 - b_1) b_0}{n} \\
 & -\left[\left(a_0 - b_0 - a_1 b_1 + b_1{}^2\right) + \frac{\left(a_0 - b_0 - a_1 b_1 + b_1{}^2\right) b_1}{n} + \frac{\left(a_0 - b_0 - a_1 b_1 + b_1{}^2\right) b_0}{n^2}\right]
\end{array}
$$

Do the subtraction.

$$\begin{array}{r l}
 & 1+\dfrac{a_1-b_1}{n}+\dfrac{a_0-b_0-a_1 b_1+b_1{}^2}{n^2}+\ldots \\
n^2+b_1 n+b_0 & \big)\, n^2+a_1 n+a_0 \\
 & -\left(n^2+b_1 n+b_0\right) \\ \hline
 & (a_1-b_1)\, n+(a_0-b_0) \\
 & -\left[(a_1-b_1)\, n+(a_1-b_1)\, b_1+\dfrac{(a_1-b_1)\, b_0}{n}\right] \\ \hline
 & \left(a_0-b_0-a_1 b_1+b_1{}^2\right)-\dfrac{(a_1-b_1)\, b_0}{n} \\
 & -\left[\left(a_0-b_0-a_1 b_1+b_1{}^2\right)+\dfrac{\left(a_0-b_0-a_1 b_1+b_1{}^2\right) b_1}{n}+\dfrac{\left(a_0-b_0-a_1 b_1+b_1{}^2\right) b_0}{n^2}\right] \\ \hline
 & -\dfrac{(a_1-b_1)\, b_0+\left(a_0-b_0-a_1 b_1+b_1{}^2\right) b_1}{n}-\dfrac{\left(a_0-b_0-a_1 b_1+b_1{}^2\right) b_0}{n^2}
\end{array}$$

Based on the new dividend, there would follow a term with $1/n^3$, a term with $1/n^4$, and so on.

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a_1 - b_1}{n} + \frac{a_0 - b_0 - a_1 b_1 + b_1^2}{n^2} + \frac{C_1}{n^3} + \frac{C_2}{n^4} + \cdots$$

$$= 1 + \frac{a_1 - b_1}{n} + \frac{a_0 - b_0 - a_1 b_1 + b_1^2 + \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots}{n^2}$$

Comparing this with equation (1), $(a_0 - b_0 - a_1 b_1 + b_1^2 + \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots)$ is bounded as n becomes large. This means that $\sum_n u_n$ converges when $a_1 - b_1 > 0$ and diverges when $a_1 - b_1 \leq 0$ by the Gauss test.